# Finite Time Estimation via Piecewise Constant Measurements ⋆

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**Abstract:** We study a broad class of linear continuous-time time-varying systems that contain piecewise continuous disturbances and piecewise constant outputs. Under an observability assumption, we construct a new type of observer to estimate the state of the system in a predetermined finite time in the presence of the disturbances. In contrast to the well-established finite time observer design techniques which estimate the system state using a continuous output, our proposed observer only requires a piecewise constant output. Our simulations illustrate the efficacy of our observer.

*Keywords:* Observer, Estimation, Finite time convergence.

## 1. INTRODUCTION

The state of the system is not available for measurement in many engineering applications such as automotive systems, bioreactors, communication systems, networked control systems, robotics, and many other fields. Instead, one aims to design an observer to estimate the state using an output that can consist of one or more, but not all, components of the state. Due to this strong motivation, many techniques for state estimation of linear continuous-time systems from output measurements, like Kalman and Luenberger observers, have been proposed in the literature; see, e.g., Kalman and Bucy (1961); Luenberger (1964); Zemouche *et al.* (2008); Ferrante *et al.* (2014).

Most of the above mentioned observer design techniques have the common disadvantage that they guarantee asymptotic convergence of the estimation error to zero, whereas it is often desirable to estimate the exact state of the system in a predetermined finite time for control and supervision purposes. Such finite time observers are of considerable interest in many applications, like in fault detection and state feedback control; see Raff and Allgower (2007); Sauvage *et al.* (2007).

Moreover, most of the observers discussed in the literature require continuous measurements. However, in many engineering applications, the measurements are piecewise constant. These systems are called continuous-discrete systems where the system dynamics are continuous while the measurements are only available at discrete instants; see Jazwinski (2007) and Ahmed-Ali *et al.* (2009) for the notion of a continuous-discrete system.

This motivates the problem of constructing finite time converging observers for systems with piecewise constant outputs. There are several works on finite time observer design for cases where the measurements are continuous instead of being piecewise constant; see, e.g., Engel and Kreisselmeier (2002); Raff and Allgower (2007); Raff and Allgower (2008); Li and Sanfelice (2015); Mazenc, Fridman and Djema (2015); Mazenc *et al.* (2017). However, to the best of our knowledge, the finite time estimation we study in this work via piecewise constant measurements has remained unsolved, even in the case of linear systems, due to the challenges of quantifying the effects of piecewise continuous disturbances on the observer performance. By contrast, for simpler cases where there are no such disturbances in the system, notable works on finite time observers include Qayyum *et al.* (2016), which uses periodic sampling times in the outputs and an observability assumption that is similar to the one we use in this work.

In the present paper, we propose a solution to the preceding problem for a family of linear continuous-time systems. We construct an observer to estimate the exact state of the system from synchronously sampled outputs. We consider a sequence of real numbers $\{t_i\}$ and a constant $\nu > 0$ such that $t_0 = 0$ and $t_{i+1} - t_i = \nu$ for all integers $i \geq 0$. Then the $t_i$'s will serve as the measurement instants for the output and $\nu$ will be a tuning parameter that will govern the estimation error. We will show that the smaller the tuning parameter $\nu$, the better the estimation. We also provide an approximate estimate of the system's state that overcomes the problem of determining explicit formulas for fundamental solutions. Our strategy has several steps. We use a classical prediction result, the finite time observer design technique of Mazenc, Fridman and Djema (2015), Mazenc *et al.* (2017), and finally a novel construction of continuous-discrete observers to complete the observer de-

⋆ This work is supported by the PHC Bosphore 2016 France-Turkey Project under project numbers 35634QM (France) and EEEAG-115E820 (TÜBİTAK- The Scientific and Technological Research Council of Turkey).

sign; see, e.g., Mazenc, Andrieu and Malisoff (2015) for the notion of continuous-discrete observer. We establish robust stability of the observer with respect to disturbances in the system dynamics.

Our paper shares fundamental features with the significant work of Shim and Teel (2003). The idea of repeatedly reconstructing the state values in a short amount of time is already present in Shim and Teel (2003), where a semi-globally stabilizing sampled output feedback for a nonlinear system is proposed. However, there are three key differences between the present paper and Shim and Teel (2003). First, in Shim and Teel (2003), the output is assumed to be known at any instant. Second, high gain observers are used in Shim and Teel (2003) to obtain approximate values of the state variable, while here we adopt a finite time reconstruction strategy. Third, although Shim and Teel (2003) covers nonlinear systems and the present paper is confined to linear systems, Shim and Teel (2003) imposes a limitation on the size of the sampling period of the feedback, while none of our results here rely on a restriction of this type. In particular, the piecewise continuous disturbances in our systems can capture the effects of sampled feedbacks with arbitrarily large sampling periods.

In Section 2 we describe our objectives in detail and present two lemmas that we will use to prove our main result in Section 3. Our illustration in Section 4 includes numerical simulations and demonstrates the utility of our theory, and in Section 5, we summarize the value added by our paper and suggest future research directions.

Throughout the sequel, the notation will be simplified whenever no confusion can arise from the context. The dimensions of our Euclidean spaces are arbitrary unless otherwise noted. The Euclidean norm in $\mathbb{R}^a$ in any dimension $a$, and the induced norm of matrices, are denoted by $|\cdot|$. Let $I$ denote the identity matrix of any dimension. Let $|\cdot|_\infty$ denote the sup norm of any matrix valued function over its entire domain, and $\exp(f)$ denotes the real valued function $e^f$ for any real valued function $f$. For any matrix valued function $\Omega : \mathbb{R} \to \mathbb{R}^{n\times n}$, let $\Phi_\Omega : \mathbb{R}\times\mathbb{R} \to \mathbb{R}^{n\times n}$ denote the function such that $\frac{\partial \Phi_\Omega}{\partial t}(t,t_0) = -\Phi_\Omega(t,t_0)\Omega(t)$ and $\Phi_\Omega(t_0,t_0) = I$ for all $t \in \mathbb{R}$ and $t_0 \in \mathbb{R}$, where $I$ is the identity matrix. Then $\Phi_\Omega$ is the inverse of the fundamental solution for the time-varying linear system $\dot{q} = \Omega(t)q$.

## 2. PROBLEM STATEMENT AND PRELIMINARIES

Our objective in this section is to construct an observer for a linear continuous-time system with a piecewise constant output such that the observer converges in predetermined finite time in the presence of a disturbance in the dynamics of the system. The observer is expressed in terms of the fundamental solution of suitable time-varying system. Since the disturbance is a general piecewise continuous function, this allows systems with a discontinuous right side which were beyond the scope of Qayyum *et al.* (2016) and other works. Then in the next section, we use ideas from this section to obtain more explicit formulas for finite time observers that do not contain the fundamental matrix and therefore may be better suited to implementations where the fundamental matrix is not available in explicit closed form.

Our systems have the form

$$\begin{cases} \dot{x}(t) = Ax(t) + \delta(t) \\ y(t) = Cx(t_i) \end{cases} \tag{1}$$

with $x$ valued in $\mathbb{R}^n$ for any $n \in \mathbb{N}$, $y$ valued in $\mathbb{R}^q$ for any $q \in \mathbb{N}$, the sampling times $t_i$ being nonnegative values for all integers $i \geq 0$, and $\delta : [0,+\infty) \to \mathbb{R}^n$ being a known bounded and piecewise continuous disturbance. We assume that $A$ and $C$ are known matrices of appropriate dimensions and the following assumption throughout this paper:

*Assumption 1.* There is a constant $\nu > 0$ such that $t_{i+1} - t_i = \nu$ for all $i \geq 0$. Also, the pair $(A, C)$ is observable. □

When Assumption 1 is satisfied, we can use (Mazenc, Fridman and Djema , 2015, Lemma 1) to find a constant $T > 0$ and a constant matrix $L$ such that with the choice $F = A + LC$, the matrix

$$M_T = e^{-AT} - e^{-FT} \tag{2}$$

is invertible and such that $T/\nu$ is an integer.

To prove our main results, we use the following two lemmas, which we prove in the appendices. The first of these lemmas is from Mazenc *et al.* (2017).

*Lemma 1.* Let $M \in \mathbb{R}^{n\times n}$ be an invertible matrix. Let $N \in \mathbb{R}^{n\times n}$ be a matrix. Let $\bar{n}$ and $\bar{m}$ be two constants such that $|M^{-1}| \leq \bar{m}$ and $|N| \leq \bar{n}$. Assume that

$$\bar{m}\bar{n} < 1 \; . \tag{3}$$

Then the matrix $M + N$ is invertible and

$$\left|(M+N)^{-1} - M^{-1}\right| \leq \frac{\bar{m}^2\bar{n}}{1-\bar{m}\bar{n}} \tag{4}$$

is satisfied. □

*Lemma 2.* Let $\mathcal{A} \in \mathbb{R}^{n\times n}$ be a constant matrix. Consider the system

$$\dot{\zeta}(t) = [\mathcal{A} + \mathcal{E}(t)]\,\zeta(t) \tag{5}$$

where $\zeta$ is valued in $\mathbb{R}^n$ and $\mathcal{E} : [0,+\infty) \to \mathbb{R}^{n\times n}$ is a bounded piecewise continuous function. Let $\phi$ denote the fundamental solution of the system (5). Then for all $t_1 \in \mathbb{R}$ and $t_2 \in \mathbb{R}$ such that $t_2 \geq t_1$, the inequality

$$\left|\phi(t_2,t_1) - e^{(t_2-t_1)\mathcal{A}}\right| \leq$$
$$|\mathcal{E}|_\infty e^{(t_2-t_1)|\mathcal{A}|}\frac{e^{2|\mathcal{A}|(t_2-t_1)}-1}{2|\mathcal{A}|}\exp\left(|\mathcal{E}|_\infty\frac{e^{2|\mathcal{A}|(t_2-t_1)}-1}{2|\mathcal{A}|}\right)$$

is satisfied. □

## 3. FINITE TIME OBSERVER DESIGN

Throughout this section, we consider the system (1) and assume that Assumption 1 is satisfied.

### 3.1 Exact Estimate

We provide an exact estimate of the system's state that converges in a predetermined finite time, using the piecewise constant function $\varphi(t) = t_i$ when $t \in [t_i, t_{i+1})$ and $i \geq 0$. Here and in what follows, all equalities and inequalities are for all $t \geq 0$, unless otherwise indicated. We have

$$\dot{x}(t) = Fx(t) + \delta(t) - Ly(t) + LC[x(\varphi(t)) - x(t)] \; .$$

We also have

$$x(\varphi(t)) = e^{A(\varphi(t)-t)}x(t) + \int_t^{\varphi(t)} e^{A(\varphi(t)-m)}\delta(m)\mathrm{d}m \; .$$

As an immediate consequence,

$$\begin{aligned}\dot{x}(t) &= [F+\mu(t)]\,x(t)+\delta(t)-Ly(t)\\ &\quad +LC\int_t^{\varphi(t)} e^{A(\varphi(t)-m)}\delta(m)\mathrm{d}m\end{aligned} \tag{6}$$

where $\mu(t) = LC\left(e^{A(\varphi(t)-t)} - I\right)$.

Let $\xi(t) = \Phi_{F+\mu}(t,0)x(t)$. Then

$$\dot{\xi}(t) = -\Phi_{F+\mu}(t,0)[F+\mu(t)]x(t) + \Phi_{F+\mu}(t,0)\dot{x}(t). \tag{7}$$

Using (6) and (7), we obtain

$$\begin{aligned}\dot{\xi}(t) &= \Phi_{F+\mu}(t,0)\left[\delta(t)-Ly(t)\right.\\ &\quad \left. +LC\int_t^{\varphi(t)} e^{A(\varphi(t)-m)}\delta(m)\mathrm{d}m\right].\end{aligned} \tag{8}$$

For any $T>0$ and $t\geq T$, we can integrate (8) over $[t-T,t]$ to obtain

$$\begin{aligned}\xi(t) &= \xi(t-T)\\ &\quad + \int_{t-T}^{t}\Phi_{F+\mu}(m,0)\left[\delta(m)-Ly(m)\right.\\ &\quad \left. +LC\int_m^{\varphi(m)} e^{A(\varphi(m)-s)}\delta(s)\mathrm{d}s\right]\mathrm{d}m\,.\end{aligned}$$

From the definition of $\xi$, and from the semigroup property of flow maps applied to the flow map $\Psi_{F+\mu}^{-1}$ of the system $\dot{q} = (F+\mu(t))q$, we deduce that

$$\begin{aligned}x(t) &= \Phi_{F+\mu}(t,0)^{-1}\Phi_{F+\mu}(t-T,0)x(t-T)\\ &\quad + \int_{t-T}^{t}\Phi_{F+\mu}(t,0)^{-1}\Phi_{F+\mu}(m,0)\\ &\quad \times\left(\delta(m)-Ly(m)\right.\\ &\quad \left. +LC\int_m^{\varphi(m)} e^{A(\varphi(m)-s)}\delta(s)\mathrm{d}s\right)dm\\ &= \Phi_{F+\mu}^{-1}(t,t-T)x(t-T)\\ &\quad + \int_{t-T}^{t}\Phi_{F+\mu}^{-1}(t,m)\left(\delta(m)-Ly(m)\right.\\ &\quad \left. +LC\int_m^{\varphi(m)} e^{A(\varphi(m)-s)}\delta(s)\mathrm{d}s\right)dm\,.\end{aligned} \tag{9}$$

Notice that (9) gives the exact value of the solution of the system (1) in a predetermined finite time $T$. In other words, the right hand side of (9) provides a finite time observer. However, finding an explicit expression for $\Phi_{F+\mu}$ may be difficult, which motivates our work in the next section.

### *3.2 Approximate Estimate*

It is often difficult to determine explicit expressions for fundamental solutions in order to estimate the system's state using (9). Our next objective is to provide an approximate estimate of the system's state that overcomes the problem of determining explicit formulas for the fundamental solutions, under our standing Assumption 1. In terms of the functions

$$\begin{aligned}\Sigma(T,\nu) &= \mathcal{G}\big(T,|LC|(e^{\nu|A|}-1)\big),\\ \mathcal{G}(T,s) &= se^{T|F|}\frac{e^{2|F|T}-1}{2|F|}\exp\left(s\frac{e^{2|F|T}-1}{2|F|}\right),\\ \bar{G}(T,\nu) &= \frac{|e^{-FT}|^2\Sigma(T,\nu)}{1-|e^{-FT}|\Sigma(T,\nu)},\\ \bar{\alpha}(T,\nu) &= \frac{|M_T^{-1}|^2\bar{G}(T,\nu)}{1-|M_T^{-1}|\bar{G}(T,\nu)},\end{aligned} \tag{10}$$

$$\begin{aligned}\bar{\beta}(T,\nu) &= \left|e^{-FT}\right|\bar{\alpha}(T,\nu)\\ &\quad + \left[|M_T^{-1}|+\bar{\alpha}(T,\nu)\right]\bar{G}(T,\nu),\end{aligned} \tag{11}$$

and

$$\bar{\gamma}(T,\nu) = \left[|M_T^{-1}|+\bar{\alpha}(T,\nu)\right]\left[\left|e^{-FT}\right|+\bar{G}(T,\nu)\right], \tag{12}$$

we prove the following result:

*Theorem 1.* Let the system (1) satisfy Assumption 1, where $A$, $B$, and $C$ are known constant matrices. Let $F$ and $T$ be such that $M_T$ as defined in (2) is invertible and such that $T/\nu$ is an integer, where the constant $\nu>0$ is such that

$$\max\left\{|e^{-FT}|\Sigma(T,\nu), |M_T^{-1}|\bar{G}(T,\nu)\right\} < 1. \tag{13}$$

Let

$$\begin{aligned}\hat{x}(t_i) &= M_T^{-1}\int_{t_i-T}^{t_i} e^{A(t_i-m-T)}\delta(m)\mathrm{d}m\\ &\quad -M_T^{-1}e^{-FT}\tilde{\mathcal{T}}(t_i,\delta,y)\end{aligned} \tag{14}$$

where

$$\begin{aligned}\tilde{\mathcal{T}}(t_i,\delta,y) &= \int_{t_i-T}^{t_i} e^{F(t_i-m)}\left(\delta(m)-Ly(m)\right.\\ &\quad \left. +\ LC\int_m^{\varphi(m)} e^{A(\varphi(m)-s)}\delta(s)\mathrm{d}s\right)\mathrm{d}m\,.\end{aligned}$$

Then

$$\begin{aligned}|x(t_i)-\hat{x}(t_i)| &\leq \bar{\alpha}(T,\nu)\int_{t_i-T}^{t_i} e^{A(t-m-T)}\delta(m)\mathrm{d}m\\ &\quad +\bar{\beta}(T,\nu)|\tilde{\mathcal{T}}(t_i,\delta,y)|\\ &\quad +\bar{\gamma}(T,\nu)\Sigma(T,\nu)\mathcal{T}_{\triangle}(t_i,\delta,y)\end{aligned}$$

holds with the choice

$$\begin{aligned}\mathcal{T}_{\triangle}(t_i,\delta,y) &= \int_{t_i-T}^{t_i}\left|\delta(m)-Ly(m)\right.\\ &\quad \left. +\ LC\int_m^{\varphi(m)} e^{A(\varphi(m)-s)}\delta(s)\mathrm{d}s\right|\mathrm{d}m\end{aligned} \tag{15}$$

for all integers $i\in\mathbb{N}$ such that $t_i>T$. □

*Proof:* Set $k=T/\nu$, which is a positive integer, by our assumptions. By integrating (1), we obtain

$$e^{-AT}x(t_i) = x(t_{i-k}) + \int_{t_i-T}^{t_i} e^{A(t_i-m-T)}\delta(m)\mathrm{d}m\,. \tag{16}$$

Using (9) and Lemma 2 (applied with $\mathcal{A}=F$ and $\mathcal{E}=\mu$), we obtain

$$x(t_i) = \left(e^{FT}+\kappa(t)\right)x(t_{i-k}) + \mathcal{T}(t_i,\delta,y) \tag{17}$$

with

$$\begin{aligned}\mathcal{T}(t_i,\delta,y) &= \int_{t_i-T}^{t_i}\Phi_{F+\mu}^{-1}(t_i,m)\left(\delta(m)-Ly(m)\right.\\ &\quad \left. +\ LC\int_m^{\varphi(m)} e^{A(\varphi(m)-s)}\delta(s)\mathrm{d}s\right)\mathrm{d}m\end{aligned}$$

and

$$|\kappa(t)| \leq \mathcal{G}(T, |\mu|_\infty). \tag{18}$$

Here and in the sequel, all equalities and inequalities should be understood to hold for all $t \geq t_i$ and all $i$ such that $t_i > T$.

Our formula $\mu(t) = LC\left(e^{A(\varphi(t)-t)} - I\right)$ gives

$$\begin{aligned} |\mu|_\infty &= \left|LC\left(e^{A(\varphi(t)-t)} - I\right)\right|_\infty \\ &\leq |LC| \left|\sum_{k=1}^{\infty} \frac{A^k(\varphi(t)-t)^k}{k!}\right|_\infty \\ &\leq |LC| \sum_{k=1}^{\infty} \frac{|A|^k |\varphi(t)-t|^k}{k!} . \end{aligned}$$

Since $|t - \varphi(t)|_\infty \leq \nu$, we deduce that

$$|\mu|_\infty \leq |LC| \left(e^{\nu|A|} - 1\right) . \tag{19}$$

Using (18) and (19), we have

$$|\kappa(t)| \leq \mathcal{G}\left(T, |LC|\left(e^{\nu|A|} - 1\right)\right) = \Sigma(T, \nu) \tag{20}$$

for all $t \geq 0$. Since our condition (13) on $\nu$ gives $|e^{-|F|T}|\Sigma(T, \nu) < 1$, we can use the inequality (20) and Lemma 1 (applied with $M = e^{FT}$ and $N = \kappa(t)$) to deduce that $e^{FT} + \kappa(t)$ is invertible for all $t$. Then (17) gives

$$\begin{aligned} &\left(e^{FT} + \kappa(t)\right)^{-1} x(t_i) \\ &= x(t_{i-k}) + \left(e^{FT} + \kappa(t)\right)^{-1} \mathcal{T}(t_i, \delta, y) . \end{aligned} \tag{21}$$

Combining (16) and (21), we obtain

$$\begin{aligned} &\left[e^{-AT} - \left(e^{FT} + \kappa(t)\right)^{-1}\right] x(t_i) = \\ &\int_{t_i-T}^{t_i} e^{A(t_i-m-T)}\delta(m)\mathrm{d}m - \left(e^{FT} + \kappa(t)\right)^{-1} \mathcal{T}(t_i, \delta, y) . \end{aligned}$$

Using the definition of $M_T$, we have

$$\begin{aligned} &[M_T + G(t, T)]\, x(t_i) = \int_{t_i-T}^{t_i} e^{A(t_i-m-T)}\delta(m)\mathrm{d}m \\ &- \left(e^{FT} + \kappa(t)\right)^{-1} \mathcal{T}(t_i, \delta, y) \end{aligned} \tag{22}$$

where $G(t, T) = e^{-FT} - \left(e^{FT} + \kappa(t)\right)^{-1}$. Lemma 1 (applied with $M = e^{FT}$ and $N = \kappa(t)$) also ensures that

$$|G(t, T)| \leq \bar{G}(T, \nu) \tag{23}$$

where $\bar{G}$ is from (10). Since $M_T$ is invertible, it follows from our condition (13) and the inequality (23) and Lemma 1 (applied with $M = M_T$, $N = G(t, T)$, and $\bar{n} = \bar{G}(T, \nu)$) that $M_T + G(t, T)$ is invertible and from (22), we have

$$\begin{aligned} x(t_i) &= [M_T + G(t, T)]^{-1} \\ &\times \int_{t_i-T}^{t_i} e^{A(t-m-T)}\delta(m)\mathrm{d}m \\ &- [M_T + G(t, T)]^{-1} \\ &\times \left(e^{FT} + \kappa(t)\right)^{-1} \mathcal{T}(t_i, \delta, y) . \end{aligned} \tag{24}$$

From (14) and (24), we deduce that

$$\begin{aligned} &|x(t_i) - \hat{x}(t_i)| \leq \beta(t, T)|\tilde{\mathcal{T}}(t_i, \delta, y)| \\ &+\alpha(t, T)\left|\int_{t_i-T}^{t_i} e^{A(t-m-T)}\delta(m)\mathrm{d}m\right| \\ &+\gamma(t, T)|\mathcal{T}(t_i, \delta, y) - \tilde{\mathcal{T}}(t_i, \delta, y)| \end{aligned} \tag{25}$$

where $\alpha(t, T) = |[M_T + G(t, T)]^{-1} - M_T^{-1}|$,

$$\begin{aligned} \beta(t, T) &= \Big|M_T^{-1} e^{-FT} \\ &\quad - [M_T + G(t, T)]^{-1}\left(e^{FT} + \kappa(t)\right)^{-1}\Big| , \end{aligned}$$

and

$$\gamma(t, T) = \left|[M_T + G(t, T)]^{-1}\left(e^{FT} + \kappa(t)\right)^{-1}\right| .$$

Lemma 1 (applied with $M = M_T$ and $N = G(t, T)$) ensures that

$$\alpha(t, T) \leq \bar{\alpha}(T, \nu) \tag{26}$$

where $\bar{\alpha}$ was defined in (10). We have

$$\begin{aligned} \beta(t, T) &= \Big|\left(M_T^{-1} - [M_T + G(t, T)]^{-1}\right) e^{-FT} \\ &\quad + [M_T + G(t, T)]^{-1} \\ &\quad \times \left(e^{-FT} - \left(e^{FT} + \kappa(t)\right)^{-1}\right)\Big| \\ &\leq \left|M_T^{-1} - [M_T + G(t, T)]^{-1}\right| \left|e^{-FT}\right| \\ &\quad + \left|[M_T + G(t, T)]^{-1}\right| \\ &\quad \times \left|e^{-FT} - \left(e^{FT} + \kappa(t)\right)^{-1}\right| \\ &\leq \bar{\alpha}(T, \nu)\left|e^{-FT}\right| \\ &\quad + \left[|M_T^{-1}| + \bar{\alpha}(T, \nu)\nu\right] \bar{G}(T, \nu) \\ &= \bar{\beta}(T, \nu) \end{aligned}$$

with $\bar{\beta}$ also as defined in (10). We also have

$$\begin{aligned} \gamma(t, T) &= \left|[M_T + G(t, T)]^{-1}\left(e^{FT} + \kappa(t)\right)^{-1}\right| \\ &\leq \left|[M_T + G(t, T)]^{-1}\right| \left|\left(e^{FT} + \kappa(t)\right)^{-1}\right| \\ &\leq \bar{\gamma}(T, \nu) \end{aligned}$$

where $\bar{\gamma}$ is also from (10). Observe that Lemma 2 gives

$$\begin{aligned} &|\mathcal{T}(t_i, \delta, y) - \tilde{\mathcal{T}}(t_i, \delta, y)| \\ &\leq \int_{t_i-T}^{t_i} \left|\Phi_{F+\mu}^{-1}(t_i, m) - e^{F(t_i-m)}\right| \\ &\quad \times \Big|\delta(m) - Ly(m) \\ &\quad + LC\int_m^{\varphi(m)} e^{A(\varphi(m)-s)}\delta(s)\mathrm{d}s\Big| \mathrm{d}m \\ &\leq \Sigma(T, \nu)\int_{t_i-T}^{t_i} \Big|\delta(m) - Ly(m) \\ &\quad + LC\int_m^{\varphi(m)} e^{A(\varphi(m)-s)}\delta(s)\mathrm{d}s\Big| \mathrm{d}m \\ &= \Sigma(T, \nu)|\mathcal{T}_\triangle(t_i, \delta, y)| \end{aligned}$$

with $\mathcal{T}_\triangle$ as defined in (15). It follows from (25)-(26) that

$$\begin{aligned} &|x(t_i) - \hat{x}(t_i)| \leq \bar{\beta}(T, \nu)|\tilde{\mathcal{T}}(t_i, \delta, y)| \\ &+\bar{\alpha}(T, \nu)\int_{t_i-T}^{t_i} e^{A(t-m-T)}\delta(m)\mathrm{d}m \\ &+\bar{\gamma}(T, \nu)\Sigma(T, \nu)\mathcal{T}_\triangle(t_i, \delta, y), \end{aligned} \tag{27}$$

which is our desired estimate. This concludes the proof.■

## 4. ILLUSTRATION

We illustrate Theorem 1 with the system

$$\dot{x}(t) = \begin{bmatrix} 0 & 0.15 \\ -0.15 & 0 \end{bmatrix} x(t) + \begin{bmatrix} d(t) \\ 0 \end{bmatrix} \tag{28}$$

where $x = (x_1, x_2)$ is valued in $\mathbb{R}^2$, $d$ is scalar valued and represents a perturbation, and the measurement is

$$y(t) = [\,0.3 \;\; 0\,]\, x\,(t_i) \tag{29}$$

where $t_i = i\nu$ for all $i \in \mathbb{N}$. One can easily check that Assumption 1 is satisfied with $C = [0.3 \;\; 0]$, $T = 6$, and that with the choice (28), we have

$$e^{At} = \begin{bmatrix} \cos(0.15t) & \sin(0.15t) \\ -\sin(0.15t) & \cos(0.15t) \end{bmatrix} \tag{30}$$

where

$$A = \begin{bmatrix} 0 & 0.15 \\ -0.15 & 0 \end{bmatrix} . \tag{31}$$

Hence, choosing

$$L = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} \;\text{ and }\; F = A + LC = \begin{bmatrix} 0 & 0.15 \\ -0.12 & 0 \end{bmatrix},$$

we obtain

$$e^{-FT} = \begin{bmatrix} \cos\left(\frac{\sqrt{35}T}{50}\right) & -\frac{\sqrt{5}}{2}\sin\left(\frac{\sqrt{35}T}{50}\right) \\ \sin\left(\frac{\sqrt{35}T}{50}\right) & \cos\left(\frac{\sqrt{35}T}{50}\right) \end{bmatrix}, \tag{32}$$

e.g., by checking that (32) has derivative $-Fe^{-FT}$ with respect to $T$. Choosing $T = 6$, we have

$$M_T = e^{-AT} - e^{-FT} = \begin{bmatrix} -0.0715 & 0.0226 \\ 0.1386 & -0.0715 \end{bmatrix}$$

which has a nonzero determinant equal to 0.0020. Then $M_T$ is invertible and

$$M_T{}^{-1} = \begin{bmatrix} -36.0244 & -11.3718 \\ -69.8228 & -36.0244 \end{bmatrix} . \tag{33}$$

Now choosing the sampling rate to be $\nu = 0.05$, one can corroborate that (13) is satisfied with $|e^{-FT}| = 1.0838$, $\Sigma(T,\nu) = 0.0094$, $|M_T{}^{-1}| = 86.9858$, and $\bar{G}(T,\nu) = 0.0111$. Therefore, we can use (33) in the formula (14) for the continuous-discrete observer from Theorem 1 for the system (28) with $t_i = 0.05i$ for all $i \in \mathbb{N}$.

To illustrate our result, Fig. 1 shows MATLAB simulation of our observer (14) for the system (28) under a piecewise continuous perturbation $d(t) = 0.5u(t)$ with initial conditions $x_1(0) = \hat{x}_1(0) = \hat{x}_2(0) = 0$, and $x_2(0) = 2$. We have also include a zoomed plot in Fig. 1 to depict that we have used a zero-order hold with $\nu = 0.05$ to construct the piecewise continuous estimate $\hat{x}_2$ from its discrete samples. The fundamental sampling rate of our simulation is 0.1 kHz. The simulation results corroborate convergence of our estimate after $T = 6$ seconds. Since our simulations show good tracking performance, they help illustrate our general theory in the special case of the system (28) with the measurement (29).

## 5. CONCLUSION

For linear continuous-time systems with a piecewise constant output, we proposed an observer of a new type, estimating the system state in a predetermined finite time in the presence of a disturbance in the dynamics of the system. It provides an exact estimate which in general is not given by an explicit formula. This led us to propose an approximate formula, which is given by an explicit formula and whose accuracy is proportional to the size of the sampling interval. We also provided an approximate estimate to overcome the problem of computing the explicit expressions of the fundamental solutions. Many extensions of our observer design we proposed are possible, pertaining for instance to the design of reduced order observers and extensions to families of globally Lipschitz nonlinear time-varying systems and asynchronous sampling.

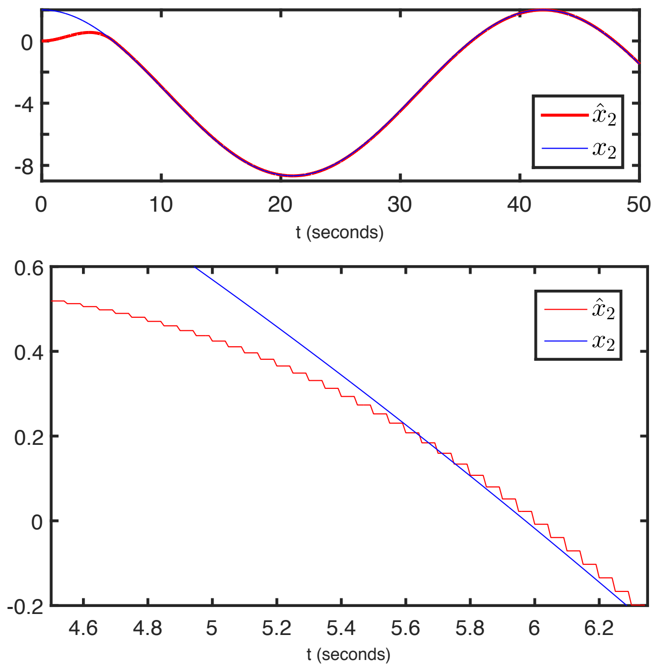


Fig. 1. Simulations of continuous-discrete observer (14) for (28): Component $x_2$ and its estimate $\hat{x}_2$

## 6. ACKNOWLEDGMENTS


The authors would like to acknowledge fruitful discussions with Professor Michael Malisoff of Department of Mathematics, Louisiana State University, Baton Rouge, USA.


## APPENDIX A. PROOFS OF LEMMAS 1 AND 2

### *A.1. Proof of Lemma 1*

To prove that the matrix $M+N$ is invertible, let us proceed by contradiction. We suppose that it is not invertible. Then there is a nonzero vector $V \in \mathbb{R}^n$ such that $V^\top(M+N) = 0$, so invertibility of $M$ gives $V^\top = -V^\top N M^{-1}$, and so also $|V| \le |V|\bar{m}\bar{n}$. Since $V \neq 0$, we conclude that $1 \le \bar{m}\bar{n}$, which contradicts (3). We deduce that $M+N$ is invertible. To prove the inequality (4), we first set $R = (M+N)^{-1} - M^{-1}$. By multiplying $R$ by $M+N$ and $M$, we obtain $(M+N)RM = M - (M+N) = -N$, and so also $MRM = -N - NRM$. We deduce that $R = -M^{-1}NM^{-1} - M^{-1}NR$. As an immediate consequence, we obtain $|R| \le \bar{m}^2\bar{n} + \bar{m}\bar{n}|R|$, which allows us to conclude the proof of Lemma 1. ■

### *A.2. Proof of Lemma 2*

Let $\phi$ be the fundamental solution of the system

$$\frac{\partial \phi}{\partial t}(t,t_0) = [\mathcal{A} + \mu(t)]\phi(t,t_0) \,. \tag{34}$$

Here and in the sequel, $t_0 \ge 0$ and $t \ge t_0$ are arbitrary. Let $\psi(t,t_0) = e^{-\mathcal{A}(t-t_0)}\phi(t,t_0)$. Then

$$\frac{\partial \psi}{\partial t}(t,t_0) = \omega(t,t_0)\psi(t,t_0) \tag{35}$$

holds with

$$\omega(t,t_0) = e^{-\mathcal{A}(t-t_0)}\mathcal{E}(t)e^{\mathcal{A}(t-t_0)}. \tag{36}$$

For any vector $V \in \mathbb{R}^n$, we have

$$\begin{aligned} &\frac{\partial}{\partial t}\left((\psi(t,t_0)V)^\top \psi(t,t_0)V\right) = \\ &V^\top \psi(t,t_0)^\top \omega(t,t_0)\psi(t,t_0)V \ . \end{aligned} \tag{37}$$

Consequently,

$$\frac{\partial(|\psi(t,t_0)V|^2)}{\partial t} \leq |\omega(t,t_0)||\psi(t,t_0)V|^2 \ . \tag{38}$$

Through a simple integration, we obtain

$$|\psi(t,t_0)V| \leq e^{\int_{t_0}^t |\omega(m,t_0)|\mathrm{d}m}|V| \ . \tag{39}$$

One can check readily that

$$|\omega(t,t_0)| \leq |\mathcal{E}|_\infty e^{2|\mathcal{A}|(t-t_0)} \ . \tag{40}$$

Consequently,

$$\begin{aligned} \int_{t_0}^t |\omega(m,t_0)|\mathrm{d}m &\leq |\mathcal{E}|_\infty \int_{t_0}^t e^{2|\mathcal{A}|(m-t_0)}\mathrm{d}m \\ &= |\mathcal{E}|_\infty \frac{e^{2|\mathcal{A}|(t-t_0)}-1}{2|\mathcal{A}|}. \end{aligned} \tag{41}$$

Combining (39) and (41), we obtain

$$|\psi(t,t_0)V| \leq \exp\left(|\mathcal{E}|_\infty \frac{e^{2|\mathcal{A}|(t-t_0)}-1}{2|\mathcal{A}|}\right)|V| \ . \tag{42}$$

Since this inequality is valid for all $V \in \mathbb{R}^n$, we have

$$|\psi(t,t_0)| \leq \exp\left(|\mathcal{E}|_\infty \frac{e^{2|\mathcal{A}|(t-t_0)}-1}{2|\mathcal{A}|}\right) \ . \tag{43}$$

Again using (35), we deduce that

$$\int_{t_0}^t \frac{\partial \psi}{\partial t}(s,t_0)\mathrm{d}s = \int_{t_0}^t \omega(s,t_0)\psi(s,t_0)\mathrm{d}s \ . \tag{44}$$

It follows from the Fundamental Theorem of Calculus that

$$\psi(t,t_0) - I = \int_{t_0}^t \omega(s,t_0)\psi(s,t_0)\mathrm{d}s \ . \tag{45}$$

We deduce that

$$\begin{aligned} &|\psi(t,t_0) - I| \leq \int_{t_0}^t |\omega(s,t_0)||\psi(s,t_0)|\mathrm{d}s \\ &\leq \int_{t_0}^t |\mathcal{E}|_\infty e^{2|\mathcal{A}|(s-t_0)}\exp\left(|\mathcal{E}|_\infty \frac{e^{2|\mathcal{A}|(s-t_0)}-1}{2|\mathcal{A}|}\right)\mathrm{d}s \end{aligned} \tag{46}$$

where the last inequality is a consequence of (43) and (40). We deduce that

$$\begin{aligned} &|\psi(t,t_0) - I| \\ &\leq |\mathcal{E}|_\infty \int_{t_0}^t e^{2|\mathcal{A}|(s-t_0)}\mathrm{d}s \exp\left(|\mathcal{E}|_\infty \frac{e^{2|\mathcal{A}|(t-t_0)}-1}{2|\mathcal{A}|}\right) \\ &= |\mathcal{E}|_\infty \frac{e^{2|\mathcal{A}|(t-t_0)}-1}{2|\mathcal{A}|}\exp\left(|\mathcal{E}|_\infty \frac{e^{2|\mathcal{A}|(t-t_0)}-1}{2|\mathcal{A}|}\right) \ . \end{aligned} \tag{47}$$

We also have

$$\begin{aligned} &\left|\phi(t,t_0) - e^{(t-t_0)\mathcal{A}}\right| \\ &= \left|e^{(t-t_0)\mathcal{A}}\left(e^{-(t-t_0)\mathcal{A}}\phi(t,t_0) - I\right)\right| \\ &\leq e^{(t-t_0)|\mathcal{A}|}\left|\psi(t,t_0) - I\right| \ . \end{aligned} \tag{48}$$

The inequality in conjunction with (47) gives

$$\begin{aligned} &\left|\phi(t,t_0) - e^{(t-t_0)\mathcal{A}}\right| \leq \\ &|\mathcal{E}|_\infty e^{(t-t_0)|\mathcal{A}|}\frac{e^{2|\mathcal{A}|(t-t_0)}-1}{2|\mathcal{A}|}\exp\left(|\mathcal{E}|_\infty \frac{e^{2|\mathcal{A}|(t-t_0)}-1}{2|\mathcal{A}|}\right) \end{aligned}$$

which is the desired conclusion. ■

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